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Unitary transformation approach to the cyclic evolution of SU(1, 1) and SU(2) time-dependent systems and geometrical phases

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Abstract. Using the time-dependent unitary transformation instead of the invariant operator, the solutions of SU(1, 1) and SU(2) time-dependent quantum systems are obtained. It is shown that the evolution operator is decomposed into the product form of two unitary operators in such a way that one of them has the same periodicity as the Hamiltonian and the other correspond to the Floquet operator which gives the cyclic states and their associated phases over one period of the evolution. The non-adiabatic Berry's (or Aharonov–Anandan) phases are determined totally by such a unitary transformation.

1. Introduction

The explicitly time-dependent quantum systems have been a long-standing mathematical problem, and are not yet completely solved in general. Various methods have been used to obtain approximate solutions for such time-dependent problems. The usual methods are the adiabatic approximation, the sudden approximation and time-dependent perturbation techniques. The existence of invariants (constants of the motion or first integral) introduced by Lewis and Riesenfeld [1] is one factor of central importance in the study of such systems. They have derived a simple relation between eigenstates of the invariant I and solutions of the time-dependent Schrödinger equation, and have applied it to the case of a quantal oscillator with time-dependent frequency. If the Hamiltonian depends explicitly on time through a set of parameters, Berry [2] has shown that, in the adiabatic hypothesis, an eigenstate of the Hamiltonian develops, besides an expected dynamical phase, a geometrical phase $\gamma_n^B(\mathcal{C})$ whose value only depends on the closed path $\mathcal{C}$ in the space of parameters. Removing the adiabatic hypothesis, Aharonov and Anandan [3] have generalized Berry's result and shown that such a phase may appear for any state which is cyclic with respect to some evolution. (Cyclicity means that the state returns to itself, after some time, up to a phase factor; in Berry's approach, the adiabatic hypothesis ensures the cyclicity of the eigenstates of $H(t)$ after one loop.) Therefore, a natural (but not unique) way to obtain such a basis of cyclic states is to consider the eigenvectors of a Hermitian periodic invariant $I(t)$. Indeed, any eigenstate of an invariant operator $I(0)$ at time zero evolves continuously into the corresponding eigenstate of the invariant operator $I(t)$ at time t [1], exactly as an eigenstate of the Hamiltonian does when the evolution is adiabatic. For this reason, invariant operators have played an important part in recent works on non-adiabatic geometric phases [4–12]. In most of these papers use is made of the Lewis–Riesenfield quadratic invariant to study two archetypal examples. One of these is the time-dependent generalized harmonic

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oscillator, the Hamiltonian of which is a time-dependent function of the SU(1, 1) generator and the other is the spin in a time-dependent varying magnetic field with Hamiltonian consisting of the SU(2) generator. In [13], the SU(1, 1) and SU(2) time-dependent systems are exactly integrated and the time evolution operator and non-adiabatic Berry's phase are obtained thanks to the invariant Hermitian operator.

In light of the above discussion, one important question motivates our work here: How could one treat SU(1, 1) and SU(2) time-dependent quantum problems and investigate the possibility of finding cyclic states and their associated phases, without making recourse to the invariant operator theory?

In this paper we answer this question from a new perspective by studying the time-dependent periodic Hamiltonian systems given by a linear combination of SU(1, 1) and SU(2) generators using a time-dependent unitary transformation. Therefore, the quantity of main importance in the study of the periodic Hamiltonian and cyclic states and the accompanying phases is the evolution operator $\mathcal{U}(t)$. In this case, one knows that the evolution operator over one period $\mathcal{U}(T)$ defines the Floquet operator F where its eigenstates are solutions of the Schrödinger equation satisfying the quantum cyclic evolution condition

$$\psi_n(t) = \mathrm{e}^{-(\imath/\hbar)\mathcal{E}_n t}\phi_n(t) \qquad \phi_n(t+T) = \phi_n(t). \tag{1.1}$$

The phase remains the sum of dynamical and geometrical phases, as in Berry's case. The eigenvalues $\{\mathcal{E}_n\}$, called 'energy bands' or 'quasi-energy' in solid-state physics, are periodic functions in the parameter space, which is the Brillouin zone [14] in the case of a spatially periodic Hamiltonian.

Because of the existence of an invariant operator, an SU(1, 1) and SU(2) time-dependent quantum system must be integrable. For an integrable system, the Hamiltonian can be transformed into a sum of time-independent commuting operators through a time-dependent unitary transformation [10, 15–18]. As a consequence, the solution of the time-dependent Schrödinger equation for an integrable system can be written as eigenstates of these commuting operators. In section 2, we obtain with the help of the appropriate unitary transformation, the solution of the SU(1, 1) and SU(2) time-dependent system, as well as the time-evolution operator. In addition, the other advantage of our approach is that the time evolution operator can be decomposed into the product form

$$\mathcal{U}(t) = V(t)R(t) \tag{1.2}$$

in such a way that the unitary operator $V(t)$ has the same periodicity as the Hamiltonian with $V(T) = V(0) = 1$, and over one period $R(t)$ correspond to the Floquet operator which gives the cyclic states and their associated phases. The non-adiabatic Berry's (or Aharonov–Anandan) phases are determined totally by such a unitary transformation.

2. Evolution of SU(1, 1) and SU(2) time-dependent systems

The SU(1, 1) and SU(2) time-dependent systems that we consider are described by the periodic Hamiltonian

$$H(t) = \omega(t)K_0 + G(t)(K_+\,\mathrm{e}^{\imath\varphi(t)} + K_-\,\mathrm{e}^{-\imath\varphi(t)}) \tag{2.1}$$

where $\omega(t)$, $G(t)$ and $\varphi(t)$ are periodic functions of time with period T. K_0 is a Hermitian operator, while $K_+ = (K_-)^+$. The commutation relations of the operators are

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = DK_0. \tag{2.2}$$

The Lie algebra of SU(2) and SU(1, 1) consists of the generators K_0 and $K_\pm$ corresponding to $D = 2$ and -2 in the commutation relations (2.2), respectively.

In a way similar to Bloch's construction for periodic crystals, one may solve the periodic time-dependent Schrödinger equation (assume $\hbar = 1$)

$$\imath \frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{2.3}$$

with the requirement that its solution looks like a generalized Bloch function, i.e. it is of the form (1.1) (the product of a wavefunction satisfying a stationary equation with a time-dependent function). Therefore, we perform the time-dependent unitary transformation identical to the unitary operator (6) of Lai *et al* [13]

$$U(t) = \exp\left\{\frac{\gamma(t)}{2}(K_+ \mathrm{e}^{-\imath\beta(t)} - K_- \mathrm{e}^{\imath\beta(t)})\right\} \tag{2.4}$$

where $\gamma(t)$ and $\beta(t)$ are arbitrary real time-dependent parameters.

Recall that under a unitary transformation $|\psi(t)\rangle \rightarrow |\tilde{\psi}(t)\rangle = U^{-1}(t)|\psi(t)\rangle$, the Hamiltonian $H(t)$ and the corresponding time evolution operator $\mathcal{U}(t)$ transform according to

$$H(t) \rightarrow \tilde{H}(t) = U^{-1}H(t)U - \imath U^{-1}\frac{\partial U}{\partial t} \tag{2.5}$$

$$\mathcal{U}(t) \rightarrow \tilde{\mathcal{U}}(t) = U^{-1}(t)\mathcal{U}(t)U(0). \tag{2.6}$$

These equations are obtained by demanding that the time evolution in the transformed frame is governed by the Schrödinger equation corresponding to the transformed Hamiltonian $\tilde{H}(t)$. Substituing (2.4) into (2.5) one can show that

$$\begin{aligned}\tilde{H}(t) = {} & K_0 D\left(\frac{\omega}{D} + (\dot{\beta}-\omega)\frac{4}{\lambda^2}\sin^2\frac{\lambda}{4}\gamma - \frac{2}{\lambda}G\sin\frac{\lambda}{2}\gamma\cos(\varphi+\beta)\right)\\ & + K_+\mathrm{e}^{-\imath\beta}\left(\frac{\omega-\dot{\beta}}{\lambda}\sin\frac{\lambda}{2}\gamma - G\cos\frac{\lambda}{2}\gamma\cos(\varphi+\beta) - \frac{\imath}{2}\{\dot{\gamma}-2G\sin(\varphi+\beta)\}\right)\\ & + K_-\mathrm{e}^{\imath\beta}\left(\frac{\omega-\dot{\beta}}{\lambda}\sin\frac{\lambda}{2}\gamma - G\cos\frac{\lambda}{2}\gamma\cos(\varphi+\beta) + \frac{\imath}{2}\{\dot{\gamma}-2G\sin(\varphi+\beta)\}\right)\end{aligned} \tag{2.7}$$

where $\lambda = \sqrt{2D}$. In the derivation of equation (2.7) use is made of the following identities:

$$\begin{aligned}U^+K_+U &= K_+\cos^2\frac{\lambda}{4}\gamma - K_-\mathrm{e}^{2\imath\beta}\sin^2\frac{\lambda}{4}\gamma - \frac{D}{\lambda}K_0\mathrm{e}^{\imath\beta}\sin\frac{\lambda}{2}\gamma\\ U^+K_-U &= K_-\cos^2\frac{\lambda}{4}\gamma - K_+\mathrm{e}^{-2\imath\beta}\sin^2\frac{\lambda}{4}\gamma - \frac{D}{\lambda}K_0\mathrm{e}^{-\imath\beta}\sin\frac{\lambda}{2}\gamma\\ U^+K_0U &= K_0\cos\frac{\lambda}{2}\gamma + \frac{1}{\lambda}(K_+\mathrm{e}^{-\imath\beta} + K_-\mathrm{e}^{\imath\beta})\sin\frac{\lambda}{2}\gamma\\ \imath U^+\frac{\partial U}{\partial t} &= -2K_0\dot{\beta}\sin^2\frac{\lambda}{4}\gamma + K_+\mathrm{e}^{-\imath\beta}\left(\imath\frac{\dot{\gamma}}{2} + \frac{\dot{\beta}}{\lambda}\sin\frac{\lambda}{2}\gamma\right) + K_-\mathrm{e}^{\imath\beta}\left(-\imath\frac{\dot{\gamma}}{2} + \frac{\dot{\beta}}{\lambda}\sin\frac{\lambda}{2}\gamma\right).\end{aligned} \tag{2.8}$$

The central idea in this procedure is to simplify the transformed Hamiltonian $\tilde{H}(t)$ governing the evolution of $|\tilde{\psi}(t)\rangle$ by cancelling the term K_+ (or K_-). This is achieved by requiring

$$\begin{aligned}&\dot{\gamma} = 2G\sin(\varphi+\beta)\\ &\frac{\omega-\dot{\beta}}{\lambda}\sin\frac{\lambda}{2}\gamma = G\cos\frac{\lambda}{2}\gamma\cos(\varphi+\beta)\end{aligned} \tag{2.9}$$

by which γ and β are detemined for given values of G, φ and ω. The important point to note here is that the auxiliary equations (2.9) appears automatically in this process and are idendical to equations (7) and (8) for Lai *et al* [13] who used the general method of Lewis and Riesenfield to derive them. Then the transformed Hamiltonian $\tilde{H}(t)$ becomes

$$\tilde{H} = g(t)K_0$$
$$g(t) = D\left(\frac{\omega}{D} + (\dot{\beta} - \omega)\frac{4}{\lambda^2}\sin^2\frac{\lambda}{4}\gamma - \frac{2}{\lambda}G\sin\frac{\lambda}{2}\gamma\cos(\varphi + \beta)\right) \tag{2.10}$$

and the time evolution operator (2.6) is obviously

$$\mathcal{U}(t,0) = \exp\left\{\frac{\gamma(t)}{2}(K_+\,\mathrm{e}^{-\imath\beta(t)} - K_-\,\mathrm{e}^{\imath\beta(t)})\right\}\exp\left[-\imath\int_0^t \mathrm{d}t'\,g(t')K_0\right]$$
$$\times\exp\left\{-\frac{\gamma_0}{2}(K_+\,\mathrm{e}^{-\imath\beta_0} - K_-\,\mathrm{e}^{\imath\beta_0})\right\} \tag{2.11}$$

and has all the properties of a unitary evolution operator. Here $\gamma_0 = \gamma(0)$ and $\beta_0 = \beta(0)$ are the initial values. The implication of these results is clear. The original time-dependent quantum problem (posed through the Hamiltonian (2.1) and related to an associated time-independent Hamiltonian multiplied by an overall time-dependent factor) is completely solved without employing the Lewis–Riesenfeld theory (or invariant operator theory) [1].

In the case when the parameters $\omega(t)$, $G(t)$ and $\varphi(t)$ are T-periodic, i.e. $(\omega, G, \varphi)(t + T) = (\omega, G, \varphi)(t)$ for some T, equations (2.9) may have periodic solutions. When the auxiliary functions are periodic, i.e. $(\gamma, \beta)(t + T) = (\gamma, \beta)(t)$, then the time evolution operator can be decomposed into the product form

$$\mathcal{U}(t) = V(t)R(t) \tag{2.12}$$

where $V(t) = U(t)\,U^{-1}(0)$ is T-periodic and unitary with $V(T) = V(0) = 1$ and

$$R(t) = U(0)\exp\left[-\imath\int_0^t \mathrm{d}t'\,g(t')K_0\right]U^{-1}(0)$$

which correspond to the Floquet operator over one period. Decomposition of this type has attracted much interest in the study of periodic system [19–21]. Replacing D by ± 2 in the above expression of the time evolution operator, we obtain the time evolution operator and the Floquet operator for the SU(2) and SU(1, 1) systems, respectively.

We will now show how this result gives the cyclic states. Let $|\phi_n\rangle$ be the eigenstate of K_0 with eigenvalue k_n, i.e.

$$K_0|\phi_n\rangle = k_n|\phi_n\rangle. \tag{2.13}$$

The cyclic states $|\psi_n(0)\rangle = U(0)|\phi_n\rangle$ for the evolution in question are precisely the eigenstates of $\mathcal{U}(T) = R(T)$ with eigenphases $\alpha_n(T) = k_n\int_0^T \mathrm{d}t'\,g(t')$, as can be seen from

$$|\psi_n(T)\rangle = \mathcal{U}(T)|\psi_n(0)\rangle = \exp\left[\imath k_n\int_0^T \mathrm{d}t'\,g(t')\right]|\psi_n(0)\rangle \tag{2.14}$$

i.e. each state returns to itself after time T (in the evolution associated with $H(t)$) up to a phase $\alpha_n(T)$. In the other words, the solutions $|\psi_n(t)\rangle$ are the Floquet (or Bloch) states with associated quasi-energy $\mathcal{E}_n = \alpha_n(T)/T$. Then by virtue of equation (2.5) the quasi-energy

spectrum can be determined as

$$\alpha_n(T) = k_n \int_0^T \mathrm{d}t\, g(t) = \int_0^T \mathrm{d}t\, g(t) \int_0^T \langle \psi_n(0)|U^{-1}(0)K_0U(0)|\psi_n(0)\rangle$$
$$= \int_0^T \langle \psi_n(t)|H(t)|\psi_n(t)\rangle\, \mathrm{d}t - \int_0^T \mathrm{d}t\, \langle \psi_n(0)|\imath V^{-1}(t)\frac{\partial}{\partial t}V(t)|\psi_n(0)\rangle$$
$$= \gamma_n^{\mathrm{d}}(T) + \gamma_n^{\mathrm{g}}(T). \tag{2.15}$$

$$\gamma_n^{\mathrm{d}}(T) = \int_0^T \mathrm{d}t\, \langle \psi_n(t)|H(t)|\psi_n(t)\rangle \tag{2.16}$$

$$\gamma_n^{\mathrm{g}}(T) = \int_0^T \langle \phi_n(t)|\imath\frac{\partial}{\partial t}|\phi_n(t)\rangle = \int_0^T \langle \phi_n|\imath U^{-1}(t)\frac{\partial}{\partial t}U(t)|\phi_n\rangle$$
$$= 2k_n \oint \sin^2\frac{\lambda}{4}\gamma\, \mathrm{d}\beta \tag{2.17}$$

where $|\phi_n(t)\rangle = V(t)|\psi_n(0)\rangle$ and where we have used the commutation relations (2.2) and equations (2.8). The phases $\gamma_n^{\mathrm{d}}(T)$ and $\gamma_n^{\mathrm{g}}(T)$ were called the dynamical and geometrical (non-adiabatic Berry's) phases [3]. Thus, $\gamma_n^{\mathrm{g}}(\mathcal{C})$ for SU(2) and SU(1, 1) systems depend on $D = \pm 2$ and have a rather interesting structure, being the product of two quite distinct parts, the expectation value of K_0 for the corresponding autonomous system and a circuit integral in parameter space.

Before concluding this paper, we give a few special examples, i.e. a spin in a time-varying magnetic field and a quantum particle moving in the potential $W(q, t) = (1/2)(X(t)q^2 + Z(t)l^2/q^2)$. First, we consider $D = 2$ with $\lambda = \pm 2$ where Hamiltonian (2.1) possesses the symmetry of the dynamical group SU(2). A spin in a time-varying magnetic field is a practical example in this case [9, 13, 16, 21, 22]. Let $K_0 = J_3$ and $K_\pm = J_\pm$. $|j, n\rangle$ are the eigenvectors of J_3 , i.e. $J_3|j, n\rangle = n|j, n\rangle$. The next step is the calculation of the phases $\alpha_n(T)$ of equation (2.15). These are given by

$$\alpha_n(T) = 2n \int_0^T \mathrm{d}t \left(\frac{\omega}{2} + (\dot{\beta} - \omega)\sin^2\frac{\gamma}{2} - G\sin\gamma\cos(\varphi + \beta)\right) \tag{2.18}$$

and the non-adiabatic Berry's phase is

$$\gamma_n^{\mathrm{g}}(T) = n \oint_{\mathcal{C}} \mathrm{d}\beta\, (1 - \cos\gamma) = n\Omega(\mathcal{C}) \tag{2.19}$$

where $\Omega(\mathcal{C})$ is the solid angle subtended by the curve $\mathcal{C}$.

Second, we consider $D = -2$ with $\lambda = \pm 2\mathrm{i}$. The SU(1, 1) Lie algebra has a realization in terms of

$$K_0 = \frac{1}{4}\left[q^2 + p^2 + \frac{l^2}{q^2}\right] \qquad K_\pm = \frac{1}{4}\left[q^2 - p^2 \mp \mathrm{i}(pq + qp) - \frac{l^2}{q^2}\right]. \tag{2.20}$$

The Hamiltonian (2.1) then describes the so-called 'singular oscillator'. Various properties and physical applications to vibrational modes of polyatomic molecules of quantum systems described by this Hamiltonian have also been studied [10, 23–26]. Substitution of $D = -2$ and $\lambda = \pm 2\mathrm{i}$ into equation (2.15) yields

$$\alpha_n(T) = -2\left(n + \frac{c}{2} + \frac{3}{4}\right)\int_0^T \mathrm{d}t \left(-\frac{\omega}{2} + (\dot{\beta} - \omega)\sinh^2\frac{\gamma}{2} - G\sinh\gamma\cos(\varphi + \beta)\right) \tag{2.21}$$

where $|n, c\rangle$ are the eigenvectors of K_0, i.e. $2K_0|n, c\rangle = 2(n + c/2 + 3/4)|n, c\rangle$ and $l^2 = c(c + 1)$. The non-adiabatic Berry's phase is

$$\gamma_n^{\mathrm{g}}(T) = -\left(n + \frac{c}{2} + \frac{3}{4}\right)\oint_{\mathcal{C}} \mathrm{d}\beta\, \cosh\gamma. \tag{2.22}$$

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